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# Introduction to Nonlinear Attitude Estimation for Aerial Robotic Systems

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## Abstract

A robust and reliable attitude estimator is a key technology enabler for the development of autonomous aerial vehicles. This paper is an introduction to attitude estimation for aerial robotic systems. First, attitude definition and parameterizations are recalled and discussed. Then, several attitude estimation techniques ranging from algebraic vector observation-based attitude determination algorithms to dynamic attitude filtering and estimation methodologies are presented and commented upon in relation to practical implementation issues. Particular attention is devoted to the applications of a well-known nonlinear attitude observer (called explicit complementary filter in the literature) to aerial robotics, using a low-cost and light-weight inertial measurement unit, which can be complemented with a GPS or airspeed sensors.

## 1 Introduction

The growing interest of the robotics research community in aerial robotic vehicles is partly related to numerous applications, such as surveillance, inspection, and mapping. The development of a small-scale low-cost autonomous aerial vehicle requires effective solutions to a number of key technological problems. The avionics system of such a vehicle is arguably the technology that is most closely coupled to the autonomy of the vehicle. Within an avionics system, the attitude estimator provides the primary measurement that ensures robust stability of the vehicle flight. The development of a robust and reliable attitude estimator that can run on low-cost computational hardware and that requires only measurements from low-cost and light-weight sensing systems, is a key technology enabler for the development of such systems.

This paper is an introduction to attitude estimation for aerial robotic systems, with a focus on nonlinear attitude observers. In fact, recent advances in observer theory have led to the development of a significant body of nonlinear attitude observers [13], [25], [26], [32], [40], [47], [51]. These observers are algorithmically simple and can be implemented on low-processing power microprocessors in unit quaternion form. They need only vector measurement inputs from low-cost and light-weight microelectromechanical system (MEMS) strap-down inertial measurement units (IMUs), which can be further complemented with a GPS or airspeed sensors. Typically, the algorithms make use of a measurement of angular velocity, measured by a 3-axis gyroscope, a vector direction estimate of the gravitational direction derived from a 3-axis accelerometer (based on the small acceleration assumption) and where possible, vector measurement of the magnetic field, measured by a 3-axis magnetometer [13], [25], [32]. All low-cost MEMS devices are subject to significant noise effects. Gyroscopes and accelerometers suffer from time-varying bias and noise due to temperature change, vibration and impacts; magnetometer readings are corrupted by onboard magnetic fields generated by motors and currents, as well as external magnetic fields experienced by vehicles that maneuver in built environments. Earlier work in the development of attitude observers tackled the question of bias in the gyrometer MEMS devices [13], [25], [32], [47], [49] by introducing an adaptive bias estimate in the algorithm. Decoupling of input signals to ensure that the roll and pitch estimates are not affected by deviations in the magnetometer measurements was considered in [17], [32], [16] and represents an important modification of the basic algorithm to improve the overall quality of the attitude estimate. On the other hand, when the vehicle is subject to important linear accelerations, the attitude estimate provided by conventional solutions can be significantly erroneous, since the vector direction estimate of the gravitational direction is no longer close to

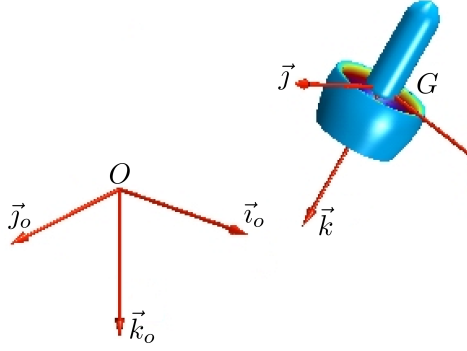


Figure 1: Attitude represents the orientation of body frame  $\mathcal{B} = \{G; \vec{i}, \vec{j}, \vec{k}\}$  w.r.t. inertial frame  $\mathcal{I} = \{O; \vec{i}_o, \vec{j}_o, \vec{k}_o\}$ .

that obtained from the accelerometer measurements. To cope with strong accelerations, a complementary GPS measurement of the vehicles linear velocity can be used to estimate its linear acceleration and, subsequently, to improve the precision of the attitude estimate [31], [15], [38]. In addition, in the case of an aircraft performing a level turn, air pressure sensors, such as pitot tubes that measure the magnitude of the airspeed, can be combined with accelerometer readings in order to derive a more precise estimate of the gravitational direction and, thus, significantly improve the quality of the attitude estimate [23].

The paper is organized as follows. In Section 2, attitude definition and parameterizations are recalled and discussed. In Section 3, existing attitude estimation methods based on vector observations, including both static and dynamics attitude estimation methodologies, are reviewed with a particular discussion on a nonlinear *explicit complementary filter/observer* [25] that was proposed by the last two authors of this paper and has become a common solution for most aerial robotic applications. Then, Section 4 presents some relevant nonlinear attitude observers for aerial robotic systems, that were developed on the basis of the explicit complementary filter. Finally, conclusions are given in Section 5.

## 2 Attitude parameterizations

The attitude represents the orientation of a frame, attached to the moving rigid body (*i.e.*, body frame  $\mathcal{B}$ ), with respect to (w.r.t.) an inertial reference frame  $\mathcal{I}$  (see Fig. 1). It can be described by a rotation matrix, an element of the special orthogonal group  $\text{SO}(3)$ , where

$$\text{SO}(3) \triangleq \{R \in \mathbb{R}^{3 \times 3} \mid \det(R) = 1, R^T R = R R^T = I_3\}.$$

By denoting such a rotation matrix as  $R \in \text{SO}(3)$ , it satisfies the following differential equation

$$\dot{R} = R S(\omega), \quad (1)$$

where  $\omega = [\omega_1, \omega_2, \omega_3]^T \in \mathbb{R}^3$  is the angular velocity vector of the body frame relative to the inertial frame, expressed in the body frame; and the notation  $S(\cdot)$  denotes the skew-symmetric matrix associated with the cross product  $\times$ , *i.e.*,  $\forall a, b \in \mathbb{R}^3$ ,  $S(a)b = a \times b$ .

Studies about the rotation group  $\text{SO}(3)$  started in the eighteenth century, and the problem of parameterization of the group of rotation of the Euclidean 3D-space has received great interest since 1776, when Euler showed that this group is a three-dimensional manifold. A rotation matrix has nine scalars components, but an element of the group of rotation can be represented by a set of less than nine parameters. Three is the minimum number of parameters required for this. However, it was shown that no three-dimensional parameterization can be 1-1 (*i.e.*, its transformation to  $\text{SO}(3)$  is a global diffeomorphism) [45]. Previously, Hopf showed in 1940 that no four-dimensional parameterization can be 1-1, and that a five-dimensional parameterization can be used to represent the rotation group in a 1-1 global manner. However, the greatest inconvenience of Hopf's five-dimensional parameterization concerns the nonlinearity of the associated differential equations [45]. On the other hand, four-dimensional parameterizations [45], [39], like the quaternions parameterization, only represent the rotation group in a 2-1 manner.

Nevertheless, although the quaternion parameterization is not 1-1, no difficulty arises for practical purposes, because the transformation of a unit quaternion to  $SO(3)$  is a local diffeomorphism everywhere. Hereafter, the Euler angles and the quaternion parameterizations are recalled and discussed.

## 2.1 Euler angles parametrization

Among many three-dimensional parameterizations [45], the Euler angles are most widely-used. Their definition depends on the problem to be solved and on the chosen systems of coordinates. A definition commonly used in the aerospace field is the Euler angles parameterization with three angles  $\phi$ ,  $\theta$  and  $\psi$  corresponding to roll, pitch and yaw respectively [45], [34]. These Euler angles allow a rotation matrix  $R$  to be factorized into a product of three matrices of rotation about three axes of the body frame as follows:

$$R = \begin{bmatrix} C\psi & -S\psi & 0 \\ S\psi & C\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\phi & -S\phi \\ 0 & S\phi & C\phi \end{bmatrix},$$

where  $S$  and  $C$  denote the  $\sin(\cdot)$  and  $\cos(\cdot)$  operators. They can be computed from the rotation matrix  $R$  as

$$\begin{cases} \phi = \text{atan2}(r_{3,2}, r_{3,3}) \\ \theta = -\text{asin}(r_{3,1}) \\ \psi = \text{atan2}(r_{2,1}, r_{1,1}) \end{cases}$$

where  $r_{i,j}$  is the component of row  $i$  and column  $j$  of  $R$ . The Euler angles kinematics satisfy [45], [34]

$$\begin{cases} \dot{\phi} = \omega_1 + S\phi T\theta \omega_2 + C\phi T\theta \omega_3 \\ \dot{\theta} = C\phi \omega_2 - S\phi \omega_3 \\ \dot{\psi} = \frac{S\phi}{C\theta} \omega_2 + \frac{C\phi}{C\theta} \omega_3 \end{cases}$$

where  $T$  denotes the  $\tan(\cdot)$  operator. If  $r_{3,1} = \pm 1$ , then  $\theta = \mp\pi/2$ , but  $\phi$  and  $\psi$  are no longer well-defined. Therefore, the Euler angles constitute a parameterization of the rotation group, except at points corresponding to  $\theta = \pm\pi/2$ . Furthermore, when  $\theta = \pm\pi/2$ ,  $\dot{\phi}$  and  $\dot{\psi}$  are not well-defined either. The problem of singularities is a weakness of the Euler angles parameterization and, as a matter of fact, of all three-dimensional parameterization techniques.

## 2.2 Unit quaternion parametrization

Compared to three-dimensional parameterizations, four-dimensional parameterizations allow singularities to be avoided. The earliest formulation of the four-dimensional parameterization, as pointed out in [39], was given by Euler in 1776. Earlier in 1775, he stated that in three dimensions, every rotation has an axis. This statement can be reformulated as follows (see *e.g.* [39], [37] for the proof)

**Euler's theorem:** *For any  $R \in SO(3)$ , there is a non-zero vector  $v \in \mathbb{R}^3$  satisfying  $Rv = v$ .*

This theorem implies that the attitude of a body can be specified in terms of a rotation by some angle about some fixed axis. It also indicates that any rotation matrix has an eigenvalue equal to one. A number of four-dimensional parameterizations can be found in the literature (see *e.g.* [39], [7]) such as the Euler parameters, the quaternion parameters, the Rodrigues parameters, and the Cayley-Klein parameters. Here, only the quaternion parameters are presented.

The quaternions were first proposed by Hamilton in 1843 [14] and further studied by Cayley and Klein. A unit quaternion has the form

$$q = s + iv_1 + jv_2 + kv_3,$$

where  $s, v_1, v_2, v_3$  are real numbers satisfying  $s^2 + v_1^2 + v_2^2 + v_3^2 = 1$ , called *constituents* of the quaternion  $q$ , and  $i, j, k$  are *imaginary units* that satisfy

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

In the literature, the unit quaternion  $q$  can be represented in a more concise way as  $q = (s, v) \in \mathbb{R} \times \mathbb{R}^3$ , where  $s \in \mathbb{R}$  is the real part of the quaternion  $q$  and  $v = [v_1, v_2, v_3]^T \in \mathbb{R}^3$  is its pure part or imaginary part. The quaternions are not commutative, but associative, and they form a group known as the quaternion group where the unit element is  $\mathbf{1} \triangleq (1, 0)$  and the quaternion product  $\star$  associated with this group is defined by

$$\begin{bmatrix} s \\ v \end{bmatrix} \star \begin{bmatrix} \bar{s} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} s\bar{s} - v^T \bar{v} \\ s\bar{v} + \bar{s}v + v \times \bar{v} \end{bmatrix}.$$

The transformed rotation matrix  $R$  is uniquely defined from the unit quaternion  $q$ , using Rodrigues' rotation formula

$$R = I_3 + 2sS(v) + 2S(v)^2.$$

On the other hand, converting a rotation matrix to a quaternion is less direct. In fact, there always exists at least one component of the unit quaternion  $q$  different from zero. Once this component is identified, the quaternion can be deduced. Note that only two values of the unit quaternion  $q$  correspond to the rotation matrix  $R$  and that they have opposed signs. For example, if  $\text{tr}(R) \neq -1$ , then

$$s = \pm \frac{1}{2} \sqrt{1 + \text{tr}(R)}, \quad S(v) = \frac{R - R^T}{4s}.$$

Finally, the quaternion kinematics are given by

$$\dot{q} = \frac{1}{2} q \star \begin{bmatrix} 0 \\ \omega \end{bmatrix}.$$

The quaternion parameterization involves four parameters (*i.e.*, only one redundant parameter) and is free of singularities. The associated differential equation is linear in  $q$ . Furthermore, the structure of the quaternion group is, by itself, of great interest.

### 3 Overview on attitude estimation based on vector observations

#### 3.1 Algebraic attitude determination

The attitude is often reconstructed from the observation of at least two non-collinear vectors. The first solution is the TRIAD algorithm, proposed by Black in 1964 [44], which algebraically computes the attitude matrix from the information in both the body frame and the inertial frame of two non-collinear unit vectors. More precisely, by denoting  $v_1^{\mathcal{I}}, v_1^{\mathcal{B}}, v_2^{\mathcal{I}}, v_2^{\mathcal{B}}$  as the vectors of coordinates, expressed in the inertial frame and the body frame respectively, of two unitary Euclidean vectors  $\vec{v}_1$  and  $\vec{v}_2$ , one has  $v_1^{\mathcal{I}} = Rv_1^{\mathcal{B}}, v_2^{\mathcal{I}} = Rv_2^{\mathcal{B}}$ , and the TRIAD algorithm provides the attitude matrix  $R$  as

$$R = \sum_{i=1}^3 s_i r_i^T = \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix} \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}^T,$$

with two orthonormal triads

$$\begin{cases} s_1 = v_1^{\mathcal{I}}, s_2 = \frac{v_1^{\mathcal{I}} \times v_2^{\mathcal{I}}}{|v_1^{\mathcal{I}} \times v_2^{\mathcal{I}}|}, s_3 = s_1 \times s_2, \\ r_1 = v_1^{\mathcal{B}}, r_2 = \frac{v_1^{\mathcal{B}} \times v_2^{\mathcal{B}}}{|v_1^{\mathcal{B}} \times v_2^{\mathcal{B}}|}, r_3 = r_1 \times r_2. \end{cases}$$

Although this algorithm is simple to implement, the resulting estimated attitude matrix, in the presence of measurement noises, is not guaranteed to remain in the rotation group  $\text{SO}(3)$ , and, thus, additional projection of the computed attitude into the group  $\text{SO}(3)$  is often required (using, for example, the Gram-Schmidt orthonormalization). Moreover, the TRIAD algorithm can only accommodate two vector observations, which may lead to difficulty in treating information when the observation of more than two vectors is available. For instance, in this case, the observation of which pair of vectors provides the best attitude estimate using the TRIAD algorithm may not be known a priori. Additionally, it does not take the relative reliability of the vector observations into account, even in the case of two vector observations. These drawbacks of the TRIAD algorithm disappear in optimal algorithms, which compute the best attitude estimate based on a cost function for which all vector observations are taken into

account simultaneously. Optimal algorithms are, however, computationally more expensive than the TRIAD algorithm. The first and also the best-known optimal attitude estimation problem is the least-square Wahba problem [52]. It consists in finding a rotation matrix  $\hat{A} \in \text{SO}(3)$  which minimizes the cost function

$$J(A) \triangleq \frac{1}{2} \sum_{i=1}^{n \geq 2} k_i |v_i^{\mathcal{B}} - A v_i^{\mathcal{T}}|^2, \quad (2)$$

where  $A$  corresponds to the transpose of the estimated attitude  $\hat{R} \in \text{SO}(3)$  (*i.e.*,  $A = \hat{R}^T$ );  $\{v_i^{\mathcal{B}}\}$  is a set of measurements of  $n (\geq 2)$  unit vectors, expressed in the body frame;  $\{v_i^{\mathcal{T}}\}$  are the corresponding unit vectors, expressed in the inertial frame; and  $\{k_i\}$  is a set of non-negative weights which can be designed based on the reliability of the corresponding measurements. Wahba's problem allows arbitrary weighting of vector observations. In [42], the author proposes the particular choice  $k_i = \sigma_i^{-2}$ , the inverse variance of the measurement  $v_i^{\mathcal{B}}$ , in order to relate Wahba's problem to Maximum Likelihood Estimation of the attitude based on an uncorrelated noise model [42]. In fact, the cost function  $J(A)$  defined by Eq. (2) can be rewritten as

$$J(A) = \frac{1}{2} \sum_{i=1}^{n \geq 2} k_i \text{tr}((v_i^{\mathcal{B}} - A v_i^{\mathcal{T}})(v_i^{\mathcal{B}} - A v_i^{\mathcal{T}})^T) = \frac{1}{2} \sum_{i=1}^{n \geq 2} k_i (|v_i^{\mathcal{T}}|^2 + |v_i^{\mathcal{B}}|^2) - \text{tr}(AB^T),$$

with  $B \triangleq \sum_{i=1}^{n \geq 2} k_i v_i^{\mathcal{B}} (v_i^{\mathcal{T}})^T$ . Therefore, the problem of finding a rotation matrix  $\hat{A}$  which minimizes  $J(A)$  is equivalent to finding a rotation matrix  $\hat{A}$  which maximizes  $\text{tr}(AB^T)$ . The first solutions to Wahba's problem, based on this observation, were proposed in 1966 by Farrell and Stuelpnagel [53] and by Wessner, Velman, Brock in the same paper<sup>1</sup>. However, these solutions, being computationally expensive, are not well suited to real-time applications. For instance, Farrell and Stuelpnagel's method requires a polar decomposition of the matrix  $B$  into a product  $B = UP$  (with  $U$  an orthogonal matrix<sup>2</sup> and  $P$  a symmetric and positive semidefinite matrix) and a diagonalization of  $P$  into  $P = WDW^T$  (with  $W$  a orthogonal matrix and  $D$  a diagonal matrix whose diagonal elements are arranged in decreasing order, *i.e.*,  $D = \text{diag}(d_1, d_2, d_3)$  with  $d_1 \geq d_2 \geq d_3$ ). The optimal rotation matrix  $\hat{A}$  is then given by

$$\hat{A} = UW \text{diag}(1, 1, \det(U)) W^T.$$

As for Wessner's solution, which is a particular case of Farrell and Stuelpnagel's solution, the optimal rotation matrix  $\hat{A}$  is calculated according to

$$\hat{A} = (B^T)^{-1} (B^T B)^{1/2}.$$

For this solution, due to the inverse of  $B^T$ , a minimum of three (non-collinear) vector observations must be available, knowing that two non-collinear vectors are sufficient for attitude reconstruction using the TRIAD algorithm. In addition, the calculation of the square root of the matrix  $B^T B$  also requires expensive computation. For example, it is necessary to diagonalize  $B^T B$  as  $B^T B = W_B D_B W_B^T$  to obtain  $(B^T B)^{1/2} = W_B D_B^{1/2} W_B^T$ .

No solution to Wahba's problem was able to replace the TRIAD algorithm in practice, until Davenport's  $q$ -method [10] and the numerical technique QUEST (QUaternion ESTimator) [41] were proposed. By using the quaternion parameterization, Davenport transformed Wahba's problem into the problem of finding the largest eigenvalue  $\lambda_{max}$  of the symmetric Davenport matrix  $K \in \mathbb{R}^{4 \times 4}$  defined by

$$K \triangleq \begin{bmatrix} C - \gamma I_3 & z \\ z^T & \gamma \end{bmatrix},$$

with  $C \triangleq B + B^T$ ,  $\gamma \triangleq \text{tr}(B)$ ,  $z \triangleq \sum_{i=1}^{n \geq 2} k_i v_i^{\mathcal{B}} \times v_i^{\mathcal{T}}$ . The optimal quaternion, corresponding to the optimal rotation matrix  $\hat{A}$  of Wahba's problem, is the normalized eigenvector  $q_{max}$  of  $K$  associated with the eigenvalue  $\lambda_{max}$ . In fact, the largest eigenvalue  $\lambda_{max}$  may be obtained by solving analytically the largest zero of the fourth-degree characteristic polynomial  $\det(K - \lambda I_4)$  [10]. However, Davenport's  $q$ -method is also computationally complex. This leads to the development of the QUEST algorithm by Shuster [41] on the basis of Davenport's  $q$ -method. QUEST consists in solving numerically the equation  $\det(K - \lambda I_4) = 0$ , or equivalently

$$\lambda^4 - (a + b)\lambda^2 - c\lambda + (ab + c\gamma - d) = 0, \quad (3)$$

with<sup>4</sup>  $a \triangleq \gamma^2 - \text{tr}(\text{adj}(C))$ ,  $b \triangleq \gamma^2 + |z|^2$ ,  $c \triangleq \det(C) + z^T C z$ ,  $d \triangleq z^T C^2 z$ . More precisely, based on Shuster's observation

<sup>1</sup>These solutions were sent to Wahba and he presented them in that paper.

<sup>2</sup> $\det(U)$  can be either 1 or  $-1$ .

<sup>3</sup>Note that any symmetric matrix is diagonalizable.

<sup>4</sup>Recall that  $\text{adj}(A)A = A\text{adj}(A) = \det(A)I_n$ , for an  $n \times n$  matrix  $A$ .

that  $\lambda_{max}$  is close to  $\lambda_o \triangleq \sum_{i=1}^{n \geq 2} k_i$ , QUEST makes use of the Newton-Raphson method to solve Eq. (3), with  $\lambda_o$  as the initial guess. It thus avoids the computation of all eigenvalues of  $K$  (*i.e.*, all solutions to Eq. (3)). QUEST is theoretically less robust than Davenport's  $q$ -method, but it is clearly faster (normally few iterations are sufficient) and has proven to be reliable in practice (*e.g.*, QUEST was implemented in the Magsat satellite in 1979). Many alternative numerical solutions for QUEST and Davenport's  $q$ -method to Wahba's problem have been proposed like, for instance, the Singular Value Decomposition (SVD), the Fast Optimal Attitude Matrix (FOAM), the Estimator of the Optimal Quaternion (ESOQ), ESOQ-1, ESOQ-2 algorithms [29]. These solutions, along with QUEST, for Wahba's problem require a trade-off between computational time and precision; for instance, the number of iterations has to be defined in advance. Additionally, their main shortcoming concerns the memoryless characteristic in the sense that information contained in measurements of past attitudes is not preserved.

### 3.2 Dynamics attitude filtering and estimation

Since a filtering algorithm is usually preferred when measurements are obtained over a range of times and especially when vector measurements are noisy, many alternative solutions to algebraic methods have been proposed. They combine the vector measurements with the kinematic equation of rotation (*i.e.*, Eq. (1)) and the angular velocity measurements. In this manner, the attitude estimation methods such as TRIAD, QUEST, SVD, FOAM, ESOQ, etc. can still be used as a preprocessor (*i.e.*, the role of an attitude sensor) for a certain number of attitude filtering methods, such as in many Kalman filters (KFs), extended Kalman filters (EKFs), or Kalman-like filters (see, *e.g.*, [11], [21], [28], [9], [3] and the references therein), or nonlinear observers (see, *e.g.*, [40], [24], [25], [47], [51]). However, this process is not a prerequisite and is loosened in many attitude filtering methods, including KFs and EKFs, as proposed in [43], [1], [13], [25], [26], [30], [50], [9], etc. This leads to simpler, faster, and (probably) more accurate methodologies. For instance, consider the filter QUEST algorithm (a recursive discrete-time Kalman-like estimator) [43]. The author proposes to calculate the estimated attitude using QUEST algorithm and by propagating and updating the matrix  $B$  (which is, itself, involved in the Davenport matrix  $K$ ) as

$$B(t_k) = \mu \Phi_{3 \times 3}(t_k, t_{k-1}) B(t_{k-1}) + \sum_{i=1}^{n_k} k_i v_i^{\mathcal{B}} (v_i^{\mathcal{I}})^T,$$

where  $\Phi_{3 \times 3}(t_k, t_{k-1}) B(t_{k-1})$  is the state transition matrix of the transpose of the rotation matrix  $R$ ,  $\mu$  is a fading memory factor and  $n_k$  is the number of vector observations at time  $t_k$ . An alternative sequential algorithm for the filter QUEST is the recursive quaternion estimator (REQUEST) [1] which propagates and updates the Davenport matrix  $K$  by

$$K(t_k) = \mu \Phi_{4 \times 4}(t_k, t_{k-1}) B(t_{k-1}) + \sum_{i=1}^{n_k} k_i K_i,$$

where  $\Phi_{4 \times 4}(t_k, t_{k-1}) B(t_{k-1})$  is the quaternion state transition matrix and  $K_i$  is the Davenport matrix for a single vector observation

$$K_i = \begin{bmatrix} v_i^{\mathcal{B}} (v_i^{\mathcal{I}})^T + v_i^{\mathcal{I}} (v_i^{\mathcal{B}})^T - ((v_i^{\mathcal{B}})^T v_i^{\mathcal{I}}) I_3 & (v_i^{\mathcal{B}} \times v_i^{\mathcal{I}}) \\ (v_i^{\mathcal{B}} \times v_i^{\mathcal{I}})^T & (v_i^{\mathcal{B}})^T v_i^{\mathcal{I}} \end{bmatrix}.$$

The main shortcoming of the filter QUEST and REQUEST algorithms concerns the fading memory factor  $\mu$  which, being arbitrarily chosen, makes these solutions *suboptimal* filters. This leads to the development of the Optimal-REQUEST algorithm [8][Ch.3] which, being essentially based on the REQUEST algorithm, further optimally calculates the fading memory factor  $\mu$  in the update stage of REQUEST according to a covariance optimization argument. Note that the filter QUEST, REQUEST, and Optimal-REQUEST algorithm, being based on QUEST, are numerical gradient methods and computationally more expensive than other modern algorithms (such as nonlinear attitude observers) due to their optimization-based nature.

Other worthy examples are nonlinear attitude observers. The earliest nonlinear observer was proposed by Salcedan [40] using the unit quaternion representation. This work has led to the development of a significant body of nonlinear attitude observers over the last fifteen years, by exploiting either the unit quaternion or the rotation matrix [40], [35], [51], [47], [24], [46], [25], [49], [32], [12], [16]. The performance of recent observers is comparable to modern nonlinear filtering techniques [9]. They generally have much stronger stability and robustness properties and are simple to tune and implement [25]. Most early nonlinear attitude observers were developed on the basis of Lyapunov analysis. However, the attitude estimation problem has also become an intuitive example for the development of recent theories on invariant observers for systems with symmetry [4], [5], [6], [25], [19], [48]. For

instance, an illustrative example concerning the nonlinear *explicit complementary filter* proposed by Mahony et al. [25] is taken. This method is basically inspired by the Luenberger observer [22] in the sense that the dynamics of the estimated attitude  $\hat{R}$  contains two parts: a main part copying the dynamics of the real attitude (*i.e.*, Eq. (1)) and an innovation part allowing the correction of the estimated attitude to the real attitude. For instance, if the gyro measurements are not affected by biases, the observer dynamics is given as

$$\dot{\hat{R}} = \hat{R}S\left(\omega + \sum_{i=1}^n k_i v_i^{\mathcal{B}} \times \hat{v}_i^{\mathcal{B}}\right), \quad (4)$$

with  $k_i$  positive constant gains designed based on the reliability of the corresponding measurements, and  $\hat{v}_i^{\mathcal{B}} \triangleq \hat{R}^T v_i^{\mathcal{T}}$ . An important issue of the attitude filtering concerns the gyro bias, leading to a complementary approach in which the gyroscopes are used to filter the vector measurements, which are in turn used to estimate the gyro bias (*e.g.*, [25], [47], [51]). By making a constant gyro bias assumption (*i.e.*,  $\omega_y = \omega + b_\omega$ , with  $\omega_y$  the gyro measurement and  $b_\omega$  the constant gyro bias vector), a complete version of the explicit complementary filter is given as [25]

$$\begin{cases} \dot{\hat{R}} &= \hat{R}S(\omega_y - \hat{b}_\omega + \sigma_R) \\ \dot{\hat{b}_\omega} &= \sigma_b \\ \sigma_R &\triangleq \sum_{i=1}^n k_i v_i^{\mathcal{B}} \times \hat{v}_i^{\mathcal{B}} \\ \sigma_b &\triangleq -k_b \sigma_R \end{cases} \quad (5)$$

with  $k_b, k_i$  positive constants. This is a continuous version; however, a discrete version can be easily derived. It can also be conveniently rewritten in quaternion form [25], [16]. Furthermore, a fast attitude estimation method from vector observations like TRIAD or QUEST can be used for a good initial attitude estimate. This observer is derived by considering the Lyapunov function candidate

$$\mathcal{L} \triangleq \sum_{i=1}^n k_i |v_i^{\mathcal{B}} - \hat{R}^T v_i^{\mathcal{T}}|^2 + \frac{2}{k_b} |b_\omega - \hat{b}_\omega|^2 \quad (6)$$

whose derivative along any solution to the error system is  $\dot{\mathcal{L}} = -2\|\mathbf{P}_a(\tilde{R}M)\|^2 \leq 0$ , with  $\tilde{R} \triangleq \hat{R}^T R$ ,  $M \triangleq \sum_{i=1}^n v_i^{\mathcal{B}}(v_i^{\mathcal{B}})^T$ ,  $\|\cdot\|$  the Frobenius norm and  $\mathbf{P}_a(\cdot)$  the skew-symmetric projection operator in square matrix space. With at least two non-collinear vector observations the estimated attitude asymptotically converges to the real one, for almost all initial conditions, even in the case of time-varying vectors (see [25], [48] for the detailed proofs). This is the strongest possible result knowing that no smooth globally asymptotically stable observers exist on  $\text{SO}(3)$  due to the topology obstruction of the rotation group [2]. In the case of a single vector observation, the solutions to this filter are still well-posed, whereas analytically reconstructing the attitude from a single vector observation is not possible. In particular, almost-global asymptotical stability is still ensured in the case of a single vector observation under the *persistent excitation* assumption (*i.e.*, if the vector's direction or the vehicle's attitude is permanently varying) [26].

**Remark 1** Interestingly, the first term in the right-hand side of the Lyapunov function  $\mathcal{L}$  defined by (6) corresponds to the cost function  $J(\hat{R}^T)$  related to Wahba's problem. Therefore, in the absence of gyro bias, the (numerical) solution to the bias-free explicit complementary filter (4) converges asymptotically/exponentially to the optimal solution to Wahba's problem for almost all initial conditions under observability conditions.

A generalized version of the explicit complementary filter (5) recently proposed by Jensen [18] is given as follows

$$\begin{cases} \dot{\hat{R}} &= \hat{R}S(\omega_y - \hat{b}_\omega + K_P \sigma) \\ \dot{\hat{b}_\omega} &= -K_I \sigma \\ \sigma &\triangleq \sum_{i=1}^n k_i v_i^{\mathcal{B}} \times \hat{v}_i^{\mathcal{B}} \end{cases} \quad (7)$$

with  $k_i$  positive constant scalar gains;  $K_P$  and  $K_I$  symmetric positive definite matrices (not necessarily constant). Clearly, this generalized observer offers a larger tuning space than the explicit complementary filter (5) while ensuring similar stability and convergence properties; and it is shown that bias-free multiplicative extended Kalman filter (MEKF) and constant-gain MEKF correspond to its particular cases [18].



## 4 Nonlinear attitude observers for aerial robotic systems

### 4.1 Sensor measurements

The most basic instrumentation embarked on an aerial robot for attitude estimation is a MEMS IMU, which can be complemented with a GPS or airspeed sensors. Assume that the IMU consists of a 3-axis gyroscope, a 3-axis accelerometer and a 3-axis magnetometer.

- The 3-axis gyroscope measures the angular velocity  $\omega$

$$\omega_y = \omega + b_\omega + \eta_\omega,$$

where  $\eta_\omega \in \mathbb{R}^3$  is the measurement noise and  $b_\omega \in \mathbb{R}^3$  denotes a constant (or slowly time-varying) gyro bias. Generally, gyroscopes are rather robust to noise and quite reliable for aerial robotics applications.

- The specific acceleration  $a_B \in \mathbb{R}^3$  expressed in the body-fixed frame  $\mathcal{B}$  is defined as  $a_B = R^T(\dot{v} - ge_3)$ , where the vehicle's acceleration expressed in the inertial frame  $\mathcal{I}$  is  $\dot{v}$ , and the gravitational acceleration expressed in the frame  $\mathcal{I}$  is  $ge_3$ , with  $e_3 = (0, 0, 1)^T$ . The 3-axis accelerometer measures this specific acceleration

$$a_y = a_B + b_a + \eta_a,$$

where  $\eta_a \in \mathbb{R}^3$  is the measurement noise and  $b_a \in \mathbb{R}^3$  denotes a bias term. Accelerometers are highly sensitive to vibration and, thus, often require significant low-pass mechanical and/or digital filtering to be usable.

- The 3-axis magnetometer measures the ambient magnetic field

$$m_y = m_B + B_m + \eta_m,$$

where  $\eta_m \in \mathbb{R}^3$  is the measurement noise,  $m_B = R^T m_{\mathcal{I}} \in \mathbb{R}^3$  is the Earth's magnetic field expressed in the body frame  $\mathcal{B}$  and  $B_m \in \mathbb{R}^3$  represents the sum of all local magnetic disturbances generated by motors and currents, as well as external magnetic fields experienced by vehicles that manoeuvre in built environments. While the magnetometer noise  $\eta_m$  is usually low, the local magnetic disturbance  $B_m$  can be significant, especially if the sensor is near the power wires of the motors.

### 4.2 Application of the explicit complementary filter for IMUs

In view of the attitude estimation survey in Section 3, most existing methods of attitude estimation make use of the measurement of at least two known non-collinear inertial vectors. Obtaining measurements of two known vectors is, however, difficult in practice and in particular for small aerial robots. In fact, while the magnetometer provides a vector measurement related to the Earth's magnetic field, the accelerometer does not directly measure the gravitational direction. This is due to the vehicle's linear acceleration involved in the specific acceleration that is measured by the accelerometer. However, most robotic vertical take-off and landing (VTOL) vehicles (such as multicopters, or ducted fans) spend a significant amount of time in near hovering or slow forward flight, with  $\dot{v} \approx 0$ ; thus, using the accelerometer as an inclinometer has been shown to be efficient in practice during this flight regime. In fact, it is known that for an ideal thrust controlled aerial vehicle, the measurement of the gravitational direction cannot be directly extracted from accelerometer measurement data [27], [33], [36]. However, VTOL robotic vehicles are subject to secondary aerodynamic forces (*e.g.*, blade flapping and induced drag) that inject low frequency information on the gravitational direction into the accelerometer measurements [27], [33]. It follows that the model  $a_B \approx -gR^T e_3$  is an effective model for vector attitude measurement in a wide range of practical systems [13], [24], [25], [32].

#### 4.2.1 Standard implementation and associated coupling issues

Once the approximation  $a_B \approx -gR^T e_3$  is made, the standard implementation of the explicit complementary filter (5) consists in defining the innovation term  $\sigma_R$  as [25]

$$\sigma_R \triangleq k_1 u_B \times \hat{u}_B + k_2 \bar{m}_B \times \hat{\bar{m}}_B, \quad (8)$$

with  $k_{1,2}$  positive constant gains,  $u_B \triangleq -a_B/g$ ,  $u_{\mathcal{I}} \triangleq e_3$ ,  $\hat{u}_B \triangleq \hat{R}^T u_{\mathcal{I}}$ ,  $\bar{m}_B \triangleq m_B/|m_{\mathcal{I}}|$ ,  $\bar{m}_{\mathcal{I}} \triangleq m_{\mathcal{I}}/|m_{\mathcal{I}}|$ ,  $\hat{\bar{m}}_B \triangleq \hat{R}^T \bar{m}_{\mathcal{I}}$ . However, it has been recognized that this standard implementation encounters some coupling issues that are well discussed in [16], [17], [30].

- Magnetic disturbances and bias influence the estimation of roll and pitch angles. In many applications especially for small-size electric motorized aerial robots, significant magnetic disturbances are almost unavoidable, leading to significant time-varying deterministic error between  $m_B$  and  $R^T m_I$ . This not only leads to large estimation errors of the yaw angle but also non-negligible errors in the roll and pitch estimation.
- The dynamics of roll, pitch and yaw estimates are highly coupled. This implies that the estimation of the yaw angle strongly affects the estimation of the roll and pitch angles. This issue can be observed by analyzing the linearized system around the system equilibrium. For the sake of simplicity, let us, for instance, neglect the gyro-bias  $b_\omega$  and the dynamics of the estimated bias  $\hat{b}_\omega$  only in this discussion. This supposition in association with Eqs. (5) and (8) yields the following dynamics of the error attitude matrix  $\tilde{R} = R\hat{R}^T$

$$\dot{\tilde{R}} = -(k_1 e_3 \times \tilde{R} e_3 + k_2 \tilde{m}_I \times \tilde{R} \tilde{m}_I) \times \tilde{R}. \quad (9)$$

Consider a first order approximation of  $\tilde{R}$  around the equilibrium  $\tilde{R} = I_3$  as  $\tilde{R} \approx I_3 + x_\times$ , with  $x \in \mathbb{R}^3$ . Note that locally the first, second and third components of  $x$  correspond, respectively, to the roll, pitch and yaw error estimates. From Eq. (9) it can easily be verified that

$$\dot{x} \approx \begin{bmatrix} -k_1 - k_2(1 - \bar{m}_{I,1}^2) & k_2 \bar{m}_{I,1} \bar{m}_{I,2} & k_2 \bar{m}_{I,1} \bar{m}_{I,3} \\ k_2 \bar{m}_{I,1} \bar{m}_{I,2} & -k_1 - k_2(1 - \bar{m}_{I,2}^2) & k_2 \bar{m}_{I,2} \bar{m}_{I,3} \\ k_2 \bar{m}_{I,1} \bar{m}_{I,3} & k_2 \bar{m}_{I,2} \bar{m}_{I,3} & -k_2(1 - \bar{m}_{I,3}^2) \end{bmatrix} x \quad (10)$$

In practice, the gravity vector and the Earth's magnetic field vector (*i.e.*,  $e_3$  and  $\bar{m}_I$ ) can be “ill-conditioned” in the sense that they are very close to each other. In such a case, the third component of  $\bar{m}_I$  is dominant to its first and second ones. For example, in France  $\bar{m}_{I,3} \approx 0.9$ . As a consequence, in view of Eq. (10) the dynamics of the roll and pitch errors (*i.e.*,  $x_1$  and  $x_2$ ) are strongly coupled with the yaw error dynamics (*i.e.*,  $x_3$ ).

- On the other hand, the ill-conditioning of the two vectors  $e_3$  and  $\bar{m}_I$  may also lead to the impossibility of finding a set of “non-high” gains  $\{k_1, k_2\}$  so as to provide the system with fast time response, bearing in mind that high gains may excessively amplify measurement noises. For discussion purposes and without loss of generality, let us, for instance, assume that  $\bar{m}_{I,2} \approx 0$  (*i.e.*,  $\bar{m}_{I,1}^2 + \bar{m}_{I,3}^2 \approx 1$ ) and  $\bar{m}_{I,3}^2 \gg \bar{m}_{I,1}^2$ . Under this approximation, it is straightforward to verify that three poles of System (10) are given by:

$$\begin{cases} \lambda_1 = -(k_1 + k_2) \\ \lambda_2 = -\frac{1}{2}(k_1 + k_2) \left( 1 + \sqrt{1 - \frac{4k_1 k_2 \bar{m}_{I,1}^2}{(k_1 + k_2)^2}} \right) \\ \lambda_3 = -\frac{1}{2}(k_1 + k_2) \left( 1 - \sqrt{1 - \frac{4k_1 k_2 \bar{m}_{I,1}^2}{(k_1 + k_2)^2}} \right) \approx -\frac{k_1 k_2 \bar{m}_{I,1}^2}{k_1 + k_2} \end{cases}$$

The pole  $\lambda_1$  is associated with the pitch dynamics, and the poles  $\lambda_2$  and  $\lambda_3$  are associated with the coupled roll and yaw dynamics. The less negative pole  $\lambda_3$ , approximated by  $-k_1 k_2 \bar{m}_{I,1}^2 / (k_1 + k_2)$ , will be very close to zero if  $k_1$  and  $k_2$  are not chosen sufficiently high, since  $\bar{m}_{I,1}^2 \ll 1$ . This leads to slow time response of the coupled roll and yaw dynamics.

#### 4.2.2 Modified solutions for overcoming coupling issues

Decoupling of input signals to ensure that the roll and pitch estimates are not influenced by magnetic disturbances was considered in [32], [30], [17], [16] and represents an important modification of the standard implementation of the explicit complementary filter (*i.e.*, Observer (5) with  $\sigma_R$  given by (8)) in order to improve the overall quality of the attitude estimate. Let us discuss these strategies.

The solution proposed by Martin and Salaun [30], [32] consists in creating another inertial vector as the cross product of the gravitational direction and the Earth's magnetic field direction, and its associated “artificial” measurement based on the cross product of accelerometer and magnetometer measurements. More precisely, the following vectors are computed

$$v_I \triangleq \frac{u_I \times m_I}{|u_I \times m_I|}, \quad v_B \triangleq \frac{u_B \times m_B}{|u_B \times m_B|}, \quad (11)$$

with  $u_B \triangleq -a_B/g$ ,  $u_I \triangleq e_3$ . Then, the explicit complementary filter (5) can be applied with the innovation term  $\sigma_R$  now defined as (compare to (8))

$$\sigma_R \triangleq k_1 u_B \times \hat{u}_B + k_2 v_B \times \hat{v}_B, \quad (12)$$

with  $k_{1,2}$  positive constant gains,  $\hat{u}_B \triangleq \hat{R}^T u_I$ ,  $\hat{v}_B \triangleq \hat{R}^T v_I$ . This solution ensures the decoupling of the roll and pitch estimates from magnetic disturbances *locally* in the general case [30] and *globally* in the presence of a constant magnetic bias in the inertial frame (as additionally proved in [17]).

Inspired by the work of Martin and Salaun, we have proposed another observer termed *conditioned observer* [16], which still takes the same form as the explicit complementary filter (5) but with the modified innovation terms  $\sigma_R$  and  $\sigma_b$  given by

$$\begin{cases} \sigma_R \triangleq k_1 u_B \times \hat{u}_B + k_2 ((v_B \times \hat{v}_B)^T \hat{u}_B) \hat{u}_B \\ \sigma_b \triangleq k_3 u_B \times \hat{u}_B + k_4 v_B \times \hat{v}_B \end{cases} \quad (13)$$

with  $k_{1,2,3,4}$  positive constant gains satisfying  $k_3 > k_4$ ,  $u_B \triangleq -a_B/g$ ,  $u_I \triangleq e_3$ , and (compare to (11) and see Fig. 2)

$$v_I \triangleq \frac{\pi_{u_I} m_I}{|\pi_{u_I} m_I|}, \quad v_B \triangleq \frac{\pi_{u_B} m_B}{|\pi_{u_B} m_B|}, \quad (14)$$

with  $\pi_x \triangleq |x|^2 I_3 - x x^T, \forall x \in \mathbb{R}^3$ , denoting the projection on the plan orthogonal to  $x$ . The conditioned observer ensures the global decoupling of the roll and pitch estimates from magnetic disturbances and also from the dynamics of yaw estimate in the general case. This decoupling property is clearly stronger than that of the previous solution. Moreover, in contrast with the standard implementation of the explicit complementary filter (5), fast convergence rate can still be achieved with *non-high gains* even in the case of ill-conditioning of the gravity and Earth's magnetic field directions [16].

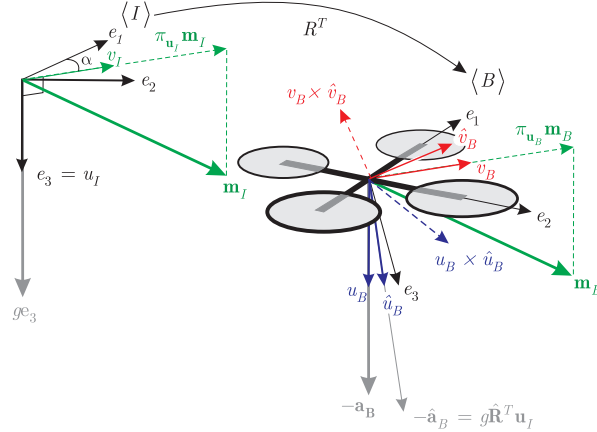


Figure 2: Vectors involved in the conditioned observer.

### 4.3 GPS-aided attitude observers

Most existing (“classical”) attitude observers/filters rely on the small acceleration assumption (*i.e.*,  $\dot{v} \ll g$ ) so that the gravitational direction measurement can be approximated by the accelerometer measurement, as discussed in the previous subsection. For many VTOL vehicles in aggressive motion, however, the vehicle's linear accelerations can be important and can induce large errors on the attitude estimate. This is also the case for fixed-wing aircraft manoeuvring in a limited space and making some rapid turns. To deal with strong linear accelerations, a complementary GPS measurement of the linear velocity can be combined with the accelerometer measurement to estimate the vehicle's acceleration and, subsequently, improve the precision of the attitude estimate. In this way, some GPS-aided attitude observers have been proposed recently [31], [15], [38] on the basis of the following differential equations

$$\begin{cases} \dot{v} &= g e_3 + R a_B \\ \dot{R} &= R S(\omega) \end{cases} \quad (15)$$

For instance, the *cascade attitude observer* proposed by Hua [15] consists in, first, estimating the specific acceleration expressed in the inertial frame  $a_{\mathcal{I}} \triangleq \dot{v} - ge_3$  and, then, in using this estimated value along with magnetometer measurements to recover the whole attitude estimate on the basis of the explicit complementary filter [25]. More precisely, in order to estimate the specific acceleration  $a_{\mathcal{I}}$ , the following observer was proposed

$$\begin{cases} \dot{\hat{v}} &= k_1(v - \hat{v}) + ge_3 + Qa_{\mathcal{B}} \\ \dot{Q} &= QS(\omega) + k_v(v - \hat{v})a_{\mathcal{B}}^T - k_q \max(0, \|Q\| - \sqrt{3})Q \end{cases} \quad (16)$$

with  $k_1, k_v, k_q$  positive constant gains and  $Q \in \mathbb{R}^{3 \times 3}$  an auxiliary matrix which is not a rotation matrix. The last term in the expression of  $\dot{Q}$  creates a dissipative effect when the Frobenius norm of  $Q$  becomes larger than  $\sqrt{3}$ , allowing it to be driven back to this threshold and thus avoiding numerical drifts of  $Q$ . It is shown that the errors  $(a_{\mathcal{I}} - Qa_{\mathcal{B}}, v - \hat{v})$  converge to zero [15]. Consequently, one can view either  $Qa_{\mathcal{B}}$  or  $Qa_{\mathcal{B}} + k_1(v - \hat{v})$  as the estimate of  $a_{\mathcal{I}}$ . From here, the author proposed the following attitude observer on the basis of the explicit complementary filter [25]

$$\begin{cases} \dot{\hat{R}} &= \hat{R}S(\omega + \sigma_R) \\ \sigma_R &\triangleq k_2 m_{\mathcal{B}} \times \hat{R}^T m_{\mathcal{I}} + k_3 a_{\mathcal{B}} \times \hat{R}^T (Qa_{\mathcal{B}} + k_1(v - \hat{v})) \end{cases} \quad (17)$$

with  $k_{2,3}$  positive constant gains. Almost global convergence of the observer is proved. Furthermore, in the special case of constant accelerations of the vehicle, almost-global asymptotic stability of the observer is achieved.

**Invariant attitude observers:** When the objective consists in combining the estimation of the attitude and the filtering of the linear velocity (and eventually the position), some invariant attitude observers have been proposed recently [31], [15], [38]. The earliest nonlinear invariant GPS-aided attitude observer was proposed by Martin and Salaun [31]. When measurements are not affected by bias, this observer has the form

$$\begin{cases} \dot{\hat{v}} &= k_1(v - \hat{v}) + ge_3 + \hat{R}a_{\mathcal{B}} \\ \dot{\hat{R}} &= \hat{R}S(\omega + \sigma_R) \\ \sigma_R &\triangleq k_2((m_{\mathcal{B}} \times \hat{R}^T m_{\mathcal{I}})^T a_{\mathcal{B}})a_{\mathcal{B}} + k_3 a_{\mathcal{B}} \times \hat{R}^T(v - \hat{v}) \end{cases} \quad (18)$$

with  $k_{1,2,3}$  positive constant gains. This defines an *invariant* observer [5], [20] in the sense that it preserves the (Lie group) invariance properties of System (15) w.r.t. constant velocity translation  $v \mapsto v + v_0$  and constant rotation of the body frame  $R \mapsto RR_0$ . A practical advantage of this solution is the (local) decoupling of the roll and pitch angles estimation from the measurements of the Earth's magnetic field (which may be rather erroneous due to magnetic disturbances). However, only local exponential stability of the estimation error is proven in [31] (based on the linearized estimation error dynamics), under some assumptions on the reference motion (*i.e.*, “smooth trajectory”).

Motivated by this result, other GPS-aided attitude invariant observers have been proposed with associated Lyapunov-based convergence and stability analyses [15], [38]. The invariant observer proposed by Hua [15] is given by

$$\begin{cases} \dot{\hat{v}} &= k_1(v - \hat{v}) + ge_3 + \hat{R}a_{\mathcal{B}} \\ \dot{\hat{R}} &= \hat{R}S(\omega + \sigma_R) \\ \sigma_R &\triangleq k_2 m_{\mathcal{B}} \times \hat{R}^T m_{\mathcal{I}} + k_3 a_{\mathcal{B}} \times \hat{R}^T(v - \hat{v}) \end{cases} \quad (19)$$

with  $k_{1,2,3}$  positive constant gains. In fact, observer (19) is slightly different from observer (18), which is a simplified version of the observer proposed in [31] suited to the case without gyro biases. The sole difference between observers (18) and (19) lies in the definition of  $\sigma_R$  where the term  $k_2((m_{\mathcal{B}} \times \hat{R}^T m_{\mathcal{I}})^T a_{\mathcal{B}})a_{\mathcal{B}}$  in (18) is replaced in (19) by  $k_2(m_{\mathcal{B}} \times \hat{R}^T m_{\mathcal{I}})$ . Another invariant observer was proposed by Robert and Tayebi [38], which can be rewritten in the following form

$$\begin{cases} \dot{\hat{v}} &= k_1(v - \hat{v}) + ge_3 + \hat{R}a_{\mathcal{B}} + \frac{1}{k_1} \hat{R}S(\sigma_R)a_{\mathcal{B}} \\ \dot{\hat{R}} &= \hat{R}S(\omega + \sigma_R) \\ \sigma_R &\triangleq k_2 m_{\mathcal{B}} \times \hat{R}^T m_{\mathcal{I}} + k_3 a_{\mathcal{B}} \times \hat{R}^T(v - \hat{v}) \end{cases} \quad (20)$$

with  $k_{1,2,3}$  positive constant gains. The additional term  $(1/k_1)\hat{R}S(\sigma_R)a_{\mathcal{B}}$  involved in the dynamics of  $\hat{v}$  in (20) constitutes the difference between observers (20) and (19), allowing the authors to establish simpler Lyapunov-based stability and convergence analyses. The main interest of both studies [15] and [38] is to yield semi-global

exponential convergence proofs. Both observers (19) and (20) guarantee the semi-global stability property under a “high-gain”-like condition on  $k_1$  which indicates that the size of basin of attraction is proportional to  $k_1$  and tends to be almost-global when  $k_1$  tends to infinity. In fact, the “high-gain” condition is only sufficient, and simulation results seem to indicate that the basin of attraction does not depend on the value of  $k_1$  ( $> 0$ ). But the proof of this property remains an open problem.

It is worth noting that, contrary to observer (18), all three observers (19), (20) and (16)–(17) do not ensure the (local) decoupling of the estimation of the roll and pitch (Euler) angles from the magnetic measurements. This suggests –as an open problem– the design of an observer that combines the advantages of these observers. For instance, observers (18) and (16)–(17) can be combined, yielding the following attitude observer (in the replacement of (17))

$$\begin{cases} \hat{\dot{R}} &= \hat{R}S(\omega + \sigma_R) \\ \sigma_R &\triangleq k_2((m_B \times \hat{R}^T m_I)^T a_B) + k_3 a_B \times \hat{R}^T (Q a_B + k_1(v - \hat{v})) \end{cases} \quad (21)$$

with  $Q$  the (numerical) solution to System (16). Specifying the stability domain of this observer, however, remains open.

#### 4.4 Airspeed-aided attitude observer for fixed-wing UAVs

For fixed-wing UAVs that manoeuvre in GPS denied environments (*e.g.*, indoor or near to buildings), an alternative solution of attitude estimation based on IMU and improved with GPS data is the use of pressure sensors such as pitot tubes that measure the magnitude of the airspeed (*i.e.*, the speed of the vehicle relative to the air) as a replacement of GPS velocity measurements. A nonlinear complementary filter/observer of this nature was proposed [23]. Magnetometer is not used in this study since the authors are only interested in roll and pitch estimation, but the incorporation of magnetometer measurements into the observer for additional yaw estimation can be done as described hereafter.

In [23], Mahony et al. consider the case where an aircraft performs a level turn (*i.e.*, constant altitude) with constant turn radius  $\rho > 0$  and zero sideslip angle. In this case, the vehicle experiences the centripetal acceleration

$$a_c \approx \omega \times (\omega \times \rho r),$$

with  $r$  the unit vector from the aircraft to the turning center. In order to eliminate the dependence on the unknown turn geometry, the approximation  $\omega \times \rho r \approx V_{\text{air}}$  is made, so that the centripetal acceleration can be approximately given by  $a_c \approx \omega \times V_{\text{air}}$ . The airspeed vector  $V_{\text{air}}$  is not directly measured, but it can be recovered from the measurement of the norm  $|V_{\text{air}}|$  given by the pitot tubes and from the knowledge of the angle-of-attack  $\alpha$  as follows

$$V_{\text{air}} = |V_{\text{air}}| \begin{bmatrix} C\alpha \\ 0 \\ S\alpha \end{bmatrix}. \quad (22)$$

The linearized dynamics model of the angle-of-attack approximately satisfy

$$\dot{\alpha} = -\frac{c_0}{|V_{\text{air}}|} \alpha + \dot{\theta} + \alpha_0, \quad (23)$$

with  $c_0$  and  $\alpha_0$  constant parameters, and  $\dot{\theta} \approx \omega_2$ . By numerically integrating Eq. (23), the angle-of-attack  $\alpha$  can be obtained, which enables the computation of the airspeed vector  $V_{\text{air}}$  according to Eq. (22) and, subsequently, of the approximated measurement of the centripetal acceleration  $a_c$  ( $\approx \omega \times V_{\text{air}}$ ).

Once the centripetal acceleration  $a_c$  is computed, the gravitational direction expressed in the body frame can also be obtained from accelerometer readings as

$$u_B = R^T u_I \approx \frac{-(a_B - a_c)}{|a_B - a_c|},$$

with  $u_I \triangleq e_3$ . Then, the explicit complementary filter (5) can be applied with the innovation term  $\sigma_R$  defined as

$$\sigma_R \triangleq k_1 u_B \times \hat{u}_B$$

with positive constant gain  $k_1$  and  $\hat{u}_B \triangleq \hat{R}^T u_I$ . Although, several assumptions and approximations are made, the reported experimental results are quite satisfactory [23].

The yaw angle may be recovered under the persistent excitation condition [26]. It can also be estimated when magnetometer measurements are involved by using the conditioned observer [16] (*i.e.*, observer (5) with the innovation terms  $\sigma_R$  and  $\sigma_b$  defined by (13)–(14)).

## 5 Conclusions

Several attitude estimation techniques –ranging from algebraic vector observations-based attitude determination algorithms to dynamics attitude filtering and estimation methodologies– have been reviewed and commented upon in relation to practical implementation issues. A particular attention is devoted to the applications of the well-known nonlinear *explicit complementary filter/observer* [25] to aerial robotics, using a low-cost and light-weight inertial measurement unit, which can be complemented with a GPS or airspeed sensors. In the case of “weak” linear accelerations, the vector direction estimate of the gravitational direction can be derived from accelerometer measurements with reasonably good accuracy and, thus, the explicit complementary filter can be directly applied. In this case, decoupling of input signals to ensure that the roll and pitch estimates are not disturbed by deviation in the magnetometer measurements represents an important improvement of the basic algorithm. On the other hand, in the case of “strong” linear accelerations, the combination of IMU with GPS-velocity or airspeed measurements allows the overall quality of the attitude estimate to be effectively improved.

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## Acronyms

- IMU: Inertial Measurement Unit.
- GPS: Global Positioning System.
- MEMS: microelectromechanical systems.
- VTOL: vertical take-off and landing.
- UAV: Unmanned Aerial Vehicle.
- QUEST: QUaternion ESTimator.
- SVD: Singular Value Decomposition.
- FOAM: Fast Optimal Attitude Matrix.
- ESOQ: Estimator of the Optimal Quaternion.

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